The Arithmetic Teacher

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THE ARITHMETIC TEACHER

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Editor: BEN A. SUELTZ, State University Teachers College, Cortland, N. Y.

Associate Editor: ESTHER J. SWENSON, University of Alabama, University, Ala.

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Principles of Learning Applied to Arithmetic

Gertrude Hildreth
Brooklyn College, New York

Children growing up in our fast-moving world are confronted with a number of developmental tasks including the learning of mathematical concepts and processes for everyday use. If children in the elementary school fail to grasp the basic principles in the use of this remarkable subject they will be left short in economic competence and many vocational opportunities will be closed to them.

Surveys of pupil competencies in elementary mathematics and a checkup of adult proficiency after a full eight years of school instruction, prove that learning arithmetic to a level of practical usefulness has not always been accomplished. In a seventh grade, the boys in an industrial arts class were puzzled as to how to cut a 15-inch board exactly in half. Likewise children in the grades come out with answers such as 5 7/8 pigs or 3 1/4 ice cream cones because they aren't certain "which process to use in problem-solving." A student handling some vital statistics knew that there were 12 per cent of a given population in the lower income group, 5 per cent in the upper income group. How many in the middle group? She could only guess since she confessed she had never been "good at percentage."

How can we account for these shortages in learning to use the basic skills of arithmetic in simple problem solving? Obviously some of the fundamental principles of learning arithmetic must have been violated or disregarded. Perhaps the children approached number work as a series of mechanical skills which were drilled in routine fashion, instead of learning to work with numbers meaningfully from the beginning. Perhaps they lacked a foundation in experiential readiness which would have prepared them for a meaningful approach to learning. It may be that computation was kept disassociated from "work with written problems" so that there never was much tie up between the two. Possibly the written problems were beyond the children's mathematical "readability" because the terms used and the language of the problems seemed a foreign language, difficult to translate. It may be that drill was actually "under done" at certain stages and that the pupils were rushed along to the next step before they had "thoroughly learned" the preceding step. More than likely all the children in the class were required to cover the same ground in the same period of time regardless of differences in learning capacity and "readiness." A teacher could have an excellent background in mathematics and yet fail in instructing children because of failure to know and apply the psychological principles of learning a skill.

Arithmetic, a Meaningful, Purposeful Experience

The child learns arithmetic best who has a rich background of relevant experience. Consequently, the teacher s first obliga-

tion is to help the children deal with numbers in items related to their own living, whatever the developmental stages they have reached. Even the seven-year olds can get the answer to the problem of the 15-inch board cut exactly in half if they have had plenty of experience using their foot rules and yardsticks in measuring.

The teacher can safeguard the pupil's future in the correct use of mathematical processes and accurate computation by spending more time teaching him to think, to reason, to generalize, instead of responding mechanically to drill, because in real life problem-solving no one ever knows what sort of problems will crop up. We want children to gain an understanding of principles, rather than mere drilled competencies in manipulating numbers so as to "get the right answer."

If children understand computation in algorism form with simple numbers they should be able to apply the process to larger, more complex numbers. If a pupil has learned the principles of fractions, e.g., the additions of fourths, perhaps he can be helped how to discover how to handle addition of sixths or eighths. The secret of "transfer" lies in grasp of the principles to be applied in situations.

There is no need to go beyond the children's daily life problems for all the applications we need to teach the fundamentals of mathematical thinking and arithmetic processes. If we teach arithmetic to children here and now, in tune with the child's mind, childish interests and understandings, their concerns, and language, so that a child fully understands what he is doing and why, in adult life he will be able to apply what he has learned to more advanced problems in mature terms of reasoning and language. This means that even in fifth grade the pupils should concentrate on their own problems such as, "We needed so many picnic plates. They come in packages of 25 or 60, at so much a package, etc.," instead of dealing with the hypothetical wholesaler with all his bales and barrels.

Learning by Experimental Trial

In teaching arithmetic we may have overstressed teaching as "teacher centered activity and neglected the child's part as an active learner, a lively experimenter, exploring number relations with concrete objects in the lower grades, and maintaining the experimental attitude, using mathematics as a process of thinking through solutions to problems in the upper grades.

The children must make their own discoveries about number relationships in order to grasp the significance of numbers and to apply arithmetic correctly in problem solving. Through experimental trials, a young child sees how 4 and 6 are related to each other, and to 10.

Trial and error is a usable concept applied to learning arithmetic if we mean by this that through his continual experimental trials in dealing with numbers the child solves the puzzles correctly with the teacher's guidance, receives "approval" from the fact that he realizes the answer is right, and is consequently all "set" for a more successful "try" the next time.

Research studies prove that young children before school age do plenty of number learning of a rudimentary sort on their own as a result of play with blocks, counting things, manipulating materials. A kindergartner discovered that he needed 4 of the small blocks to fit the space occupied by one of the long blocks or two of the half-size blocks. It must be so, because he tests it out with various combinations of blocks over and over. There is no reason why this self-learning process should not be continuous straight through the grades.

Don't discourage the child's own unique solution to a problem no matter how round-about it may seem. This is a sign that he is really comprehending mathematical processes. It is a safe conjecture that children will use all the intelligence they possess if teachers make it possible for them to exercise their thinking powers.

on lively, interesting problems related to their concerns and immediate experiencing.

Arithmetic, a New Language

Language is the children's tool of thinking in arithmetic, but it may only be a hindrance if the terms are unintelligible. Too often a language problem is created for children by the artificial phrasing of problems and the teacher's use of unfamiliar number terms. Psychologists are coming to believe that a child comprehends little or nothing of what he hears or reads that he could not say himself under ordinary circumstances. He may lisp or articulate indistinctly while saying it, but at least he knows what he is trying to say. Applied to arithmetic, this would mean that a child probably does not understand anything, no matter how "well taught," that he cannot explain in his own words or demonstrate through his own actions in some way. "2 plus 5 equals 7" may be meaningless parroting, not understood until the child recognizes that he is combining two numbers that give a larger amount when combined.

If pupils cannot explain "losing something makes less, leaves a smaller number" they have not caught the significance of *subtraction*. In their routine work with arithmetic computation the pupils may soon forget if they ever knew the origin of their term "zinto" or "guzinto" applied to certain types of problems.

Here are some other illustrations of children's difficulties with arithmetic language:

"Fewer" doll dishes was a term hard for some third grade pupils to grasp.

"How many inches in a foot? the teacher asked the new class briskly. "That depends on how long your foot is," came the snappy reply.

"Square feet." "How funny it would be if we all had square feet," was the comment of a fifth grade class on hearing this term.

"Product" has two meanings that are different in the subjects of arithmetic and geography.

"Annex two zeros" was a term that confused a sixth grader trying to use a new work book.

Commented one upper grade teacher, "I believe the children really learn better from each other, because they use their own familiar terms."

Attitudes and the Interest Factor in Learning

Mathematics could be one of the most interesting, meaningful and exciting areas in the elementary school curriculum. Artificial motivation to "get arithmetic learned" should be largely superfluous in a classroom where live wire projects are going on.

The child's attitudes are definitely tied in with success or failure in learning the "Three R's." Fear and anxiety have inhibiting effects on learning. Confidence, feelings of security, knowledge that honest effort will be recognized have the opposite effect. You cannot teach a child anything he refuses to learn because he sees no point in it. Every child must feel success in what he undertakes as motivation for continued effort to learn. He must see that his efforts count for something and he must be able to reach some concrete goal as a result of his effort.

In the learning of any complex skill plateaus will be reached. These are the discouraging impasses where difficulties pile up too fast to be easily surmounted, or discouragement sets in for various reasons. The test of a real teacher is his success in helping a child over and beyond these plateaus. What happens at the plateau may be the deciding factor in the child's desire to continue trying or to turn aside and give up.

The Role of Practice

Learning to use arithmetic requires building a series of habits, just as in the case of reading. These habits are estab-

lished only through repetition, drill and practice. As a result of practice there is steady gain in speed and efficiency because errors are eliminated and correct responses become automatic.

Use of arithmetic requires a certain amount of what the psychologists call "overlearning," that is, drill beyond the point at which the pupil has demonstrated that he can use the new process, e.g., in "borrowing" and "carrying."

In general, teachers underestimate the amount of repetition slow learners and duller children require to fix facts in mind and to habituate processes such as arithmetic requires.

Individual Differences

Varying rates of progress show up whenever children of similar age and school level try to learn the same skills. These differences are inherent in the nature of child experiences and learning capacities. Where one repetition of a fact will suffice for Child A, Child X will require a dozen repetitions and still be shaky on the fact or process. The facts about individual differences invalidate all learning programs that require the same performance standards of all at stated intervals. The same time schedule for all will mean that the slower learner has only vague notions of numbers and fails to gain mastery of processes. There is no saving in time gained through rushing ahead of the children's demonstrated level of mental development and understanding. This matter becomes particularly serious in crowded classrooms where children come from widely diverse home and educational backgrounds.

Group the pupils so that the slow can learn thoroughly and meaningfully, instead of hurriedly and superficially, without understanding. The gifted should be moving ahead with the broader economic, social and scientific applications of arithmetic.

"Conditioning" for the Use of Abstract Numbers

Success in handling number computation in standard algorism form is dependent upon understanding the meanings behind abstract numbers. In moving from number work with concrete objects and representational material to dealing with abstract numbers alone, either in print and writing or in "mental" computation, a conditioning process takes place. The meanings formed through work with a number of objects or their representations become shifted through association to the abstract number symbols. The steps in this conditioning process may be illustrated as follows:

Step 1.	(a) Real objects and representative materials, concrete experiences	→	Number meanings, number terms, computation, and solution of problems
Step 2.	(b) Graphic symbols (a) Real objects and representative materials	→	Number meanings, number terms, computation, and solution of problems
Step 3.	(b) Graphic symbols alone in computation; complete abstraction	→	Number meanings, number terms, computation, and solution of problems

By way of illustration of Step 2, suppose the problem is, "I bought x costing 12 cents and y costing 25 cents, and gave the clerk a fifty cent piece. How much money did I have left over (get back)?"

The child first acts out this problem or he sees it solved at the grocery counter. He hears the numbers used and sees them printed on the sales slip. He knows they stand for the number of cents each of the two objects costs. In the final Step 3, the pupil can now compute for solving the

problem: $50-(12+25)=13$ without handling the money or seeing the objects.

Just how to make the transition from concrete to abstract, and to determine the psychological time to accomplish it are questions for every teacher to consider. The error made in former years was to omit the first and second steps pretty largely and to begin drill with the third step. Dealing with abstract symbols is difficult for children whose experiences at the age of entering school have been entirely in the realm of concrete manipulation of objects or pictorial representation of experiences. Learning arithmetic has been made difficult for children by starting off with Step 3 instead of carefully building up to Step 3 through Steps 1 and 2, for example, in work with fractions. We must be careful not to go to the other extreme and delay the conditioning process too long, so that children are too slow in advancing to Step 3. Another mistake would be to fail to return to Step 2 after moving on to Step 3 when this would clear up difficulties in problem solving.

Grade III is about the right level to begin to make the shift from work with representative material to abstract number symbols, with these precautions: the transition should not be made too abruptly; some children are ready to switch over ahead of others. From this point on computation with abstract symbols can be rapidly learned, and must be thoroughly drilled during the remaining elementary school years.

Even after the pupil has learned to use number symbols alone in solving problems, go back to the use of manipulative and visual materials as needed. In the case of fractions this means bringing on the fruit, the measuring cups, the foot rulers, the fractions disks and "pies" cut up in equal-sized parts. When any new process is introduced demonstrate the new meanings with manipulative and visual materials, e.g., place value in decimals.

In schools where these psychological learning principles are adhered to in teaching arithmetic, the results are bearing fruit. Children are more interested in the work, they can do problem-solving earlier and with better results than ever before, they can use arithmetic with fuller understanding of what they are doing.

BOOK REVIEWS

Emerging Practices in Mathematics Education, Twenty-Second Yearbook of the National Council of Teachers of Mathematics, 1954. 434 pages, $4.50, $3.50 to members of the National Council.

As the title implies, this book contains the successful practices of sixty-one different teachers of mathematics. The scope of the yearbook is evident from its major divisions:

Part One—Various Provisions for Differentiated Mathematics Curriculums
Part Two—Laboratory Teaching in Mathematics
Part Three—Teacher Education
Part Four—New Emphases in Subject Matter
Part Five—The Evaluation of Mathematical Learning
Part Six—Bibliography of "What is Going on in Your Schools?"

While much of this book deals with topics of particular interest to teachers above the elementary school, several sections are directed specifically to the teaching of arithmetic. For example, "Handmade Materials for Teaching Arithmetic," "Preparation of Elementary Arithmetic Teachers," "An Experiment in Clinical Procedures for Arithmetic," and "Developmental Mathematics (Arithmetic) in New York City." A number of other topics serve teachers at all levels, e.g., "Some Suggestions for the use of Television in Teaching Mathematics" and "Evaluation of Mathematical Meanings and Understandings."

This is a book written by teachers for teachers. It contains much of the best practice in the United States and it is written in a straight-forward and easy-to-read style. It is full of ideas which any teacher of arithmetic and mathematics will find helpful. Dr. Clark and the committee of the National Council are to be commended for bringing the experiences of many teachers to the attention of all of us. But the yearbook is not merely a listing of experience; it is far more, for it presents guiding principles and evaluations as well.

BEN A. SUELTZ

The Use of Crutches in Teaching Arithmetic

John R. Clark[1]
New Hope, Pennsylvania

A crutch is a support or aid, for temporary use, to provide security for the user. With the help of an arithmetic crutch the pupil is expected to learn more effectively, to secure better understanding and to acquire increased facility in working with numbers.

We cannot think intelligently about the uses of crutches in arithmetic without awareness of the nature of arithmetic learning. Arithmetic learning is a composite or integration of reasoning, concepts, and techniques (skills). Our chief concern is that the pupil learn to reason, to solve problems, to be resourceful in quantitative thinking. But one cannot reason without the concepts of combining, separating, and comparing. And most problems significant to pupils require numerical answers, calling for the performance of one or more of the fundamental operations.

Many teachers are distressed by the "counting-on-the-finger" crutch. To find the sum of 7 and 4 the "finger counter" looks at one after another of four fingers and thinks eight, nine, ten, eleven. The pupil properly uses the concept of addition as "counting on," but does so immaturely. He has not learned a more mature way of thinking to find the sum, such as:

(a) breaking up or separating the four into two and two, making it likely that he can think seven and two more are nine and two more are eleven, or

(b) seven and four are as many as ten and one, or

(c) four and seven are four and four (eight) and three (eleven).

Thus finger counting indicates inadequate growth or *immaturity in thinking.* Experience in counting by groups, as well as by ones, was not adequately provided. Pupils abandon finger counting as soon as they learn better (more economical) ways of thinking. Practice on the ways of thinking in (a), (b), and (c) helps the pupils learn to recall immediately, without the use of crutches or other intermediate steps in thinking, that the sum of seven and four is eleven.

Levels or Types of Crutches

One further consideration of the nature of learning is essential. Pupils in any age group vary greatly in ability to reason, to get meanings and to manage the techniques of computation. In no third grade (eight-year old group) will all pupils have reached the same level of maturity in dealing with a system of abstractions such as is arithmetic. To find the number of 4-cent balloons one can buy with 48 cents a pupil might proceed as follows, depending upon the level of arithmetic maturity he may have reached:

Pupil A (immature): Obtain 48 pennies, arrange them in groups of four pennies each, and count the number of groups. If pennies are not available, he may use discs, or a bead frame. In any case the pupil manipulates things (uses crutches).

Pupil B (more mature): Ten balloons would cost 10×4 cents or 40 cents. With 8 cents more I could buy 2 balloons; in all, 12 balloons. (No crutch necessary.)

Pupil C (very mature): "I divide 48 cents by 4 cents, this way:

[1] Professor Clark was formerly head of the department of teaching of mathematics at Teachers College, Columbia University.

$$\begin{array}{r} \text{12 times} \\ \text{4 cents}\,\overline{)\,\text{48 cents.}} \end{array}$$

I can get 12 balloons." He is able correctly to use a standard division algorism. He needs neither crutches, nor intermediate "thinking steps."

Obviously Pupil A is not yet ready (arithmetically mature enough) to work on the level of Pupil C. Thus the kind of support or aid or crutch needed is determined by the arithmetic level of maturity of the learner. It is obvious that Pupil C would be bored and retarded if his teacher required him to work on the maturity level of Pupil A. Equally obvious, too, Pupil A should be encouraged to learn to think with symbols, so that he will be able to outgrow or be liberated from his dependence upon things (his crutches.)

The teacher selects the crutch which best helps the pupil or the class build a concept or understand a technique. To build the concept of equivalent fractions she finds "fraction-cut-outs" or "pie charts" effective. With their pie-charts the pupils *discover* that 2 fourths will exactly cover half, that 1 third is as large as 2 sixths, that 6 eighths can be exchanged for 3 fourths, etc. (The pupil also uses the charts to *demonstrate* his generalizations.)

During their use of the charts the teacher will be encouraging the pupils to think about equivalence of fractions without the aid of the charts. She guides them to make larger generalizations such as:

"If you double the number of pieces (the numerator) you make the size of the pieces (the denominator) half as large, i.e. $\frac{3}{4}=\frac{6}{8}$.

"If you make the number of pieces half as large you double the size of the pieces, i.e. $\frac{6}{8}=\frac{3}{4}$."

And finally, "A fraction may be converted into an equivalent fraction by multiplying (or dividing) its numerator and denominator by the same number."

All pupils should start with the cut-outs. Some children will soon be able to replace them with one of the generalizations just stated. Others, less mature or with less aptitude, may need to work with the cut-outs for months.

Other illustrations of physical, concrete, manipulative type of crutches are pocket charts, ten-tens frames, abacii, and measuring instruments. We shall not discuss these further, except to point out that they are effective in building concepts of base, place value, combining, separating and comparison. Their selection and use should be determined by the needs of the maturity level of the learner. The teacher has to learn when they are needed, and when they can be discarded. Use them when they support or aid the learner; throw them away when they retard the growth of the learner.

A second type of crutch employs number symbols and relates to their arrangements in written computation, i.e., to the algorisms of arithmetic. This second type includes, among others, such algorisms as the following:

For addition:

A	B	C	D	E
27	27	27→2 tens 7 ones	27	27
+15	+15	+15→1 ten 5 ones	+15	+15
12	30	3 tens 12 ones = 4 tens 2 ones	42	42
30	12			No crutch
42	42			

For subtraction:

A

$$42-15=42-10-5$$
$$=27$$

B

$$\begin{array}{rl} 4 \text{ tens } 2 \text{ ones} \rightarrow & 3 \text{ tens } 12 \text{ ones} \\ -1 \text{ ten } \ 5 \text{ ones} \rightarrow & 1 \text{ ten } \ \ 5 \text{ ones} \\ \hline & 2 \text{ tens } \ 7 \text{ ones} = 27 \end{array}$$

C

$$\begin{array}{rrr} & 3 & 12 \\ & \cancel{4} & \cancel{2} \\ - & 1 & 5 \\ \hline & 2 & 7 \end{array}$$

D

$$\begin{array}{r} 42 \\ -15 \\ \hline 27 \end{array}$$

No crutch

For multiplication:

A

$$\begin{array}{r} 24 \\ \times 15 \\ \hline 120 \\ 240 \\ \hline 360 \end{array}$$

B

$$\begin{array}{r} 24 \\ \times 15 \\ \hline 120 \\ 24 \\ \hline 360 \end{array}$$

no crutch

For division:

A

$$6\overline{)72}=6\overline{)60}^{\,10}+6\overline{)12}^{\,2}=12$$

B

$$\begin{array}{r} 2 \\ 10 \end{array}\Big\} 12$$
$$\begin{array}{r} 6\overline{)72} \\ 60 \\ \hline 12 \\ 12 \\ \hline \end{array}$$

C

$$\begin{array}{r} 12 \\ 6\overline{)72} \\ 6 \\ \hline 12 \\ 12 \\ \hline \end{array}$$

no crutch

Examination of the five algorisms for the addition example shows a progressive development from a complete recording to the minimum recording of the thinking. Algorisms A and B show the partial sums, which are combined without exchange or carrying. Algorism C emphasizes the exchange of 10 ones for 1 ten. Algorism D, with the auxiliary figure one, emphasizes the one ten resulting from the 12 ones (the 1 ten to be carried). The merit of algorism E is its brevity and economy. Obviously the first four algorisms may profitably be used to lead to or get ready for algorism E, the so-called standard addition algorism. When so used, they are effective crutches or aids.

Many teachers have objected to algorism D, arguing that once they see it, pupils persist in using it. Such is not the experience, however, of teachers who encourage the pupils to evaluate a variety of related algorisms, rating them for meaning and for economy. Good teachers are quite content for an immature learner to use relatively crude algorisms, i.e., to choose algorisms in terms of the maturity of the learner.

Now we refer to the four subtraction algorisms previously stated. Again, we note their progressive development. In A the subtrahend is separated into 10 and 5, 10 from 42 is 32; 5 from 32 is 27. There is no problem of borrowing or exchanging. In B the changing of 1 of the 4 tens to 10 ones is featured. In C we use the auxiliary minuend figures, written above the crossed out minuend figures. (Algorisms B and C are very closely related.) In D only the minimum record is made, only the result of the thinking. Algorism D certainly represents a higher level of maturity. It represents the final, not the initial form of response to be expected of the learner. Algorisms B and C obviously lead to or prepare for Algorism D. No teacher who believes in meaningful learning should hesitate to use forms B and C in getting ready for form D.

The two multiplication algorisms are chosen to illustrate the meaning of the placement of the second partial product in the standard algorism B. Algorism A clearly shows the 5×24 (120) and the 10×24 (240), and thus is a good crutch to prepare for algorism B. Most pupils willingly discontinue the use of algorism A when they see why the zero need not be written after the 24 in the second partial product. Likewise they omit writing the zeros shown in the second partial product here:

$$\begin{array}{r} 235 \\ \times 406 \\ \hline 1410 \\ 000 \\ 90 \\ \hline 95410 \end{array}$$

Three algorisms are shown for finding the quotient of 72 divided by 6 (a division beyond the range of basic division facts). In algorism A the dividend is separated into two parts (60 and 12) and each part is then divided by the divisor. In B the partial quotients are recorded vertically rather than horizontally as in A. In C we have the standard algorism for long division, which obviously is a condensed version of algorism B. Again the progressive refinement and brevity of algorisms is apparent. Algorism C is the end product, the mature final form.

The Newer Point of View

Before concluding our discussion of algorisms we call attention to a radical change in point of view concerning them. Until recently we heard a great deal about *rationalizing* (making meaningful and sensible) the algorisms of computational techniques. Then the teacher, starting with a final standard algorism, showed or attempted to show, that it was valid. She demonstrated its reasonableness and demanded its use. Today the teacher encourages pupil to devise algorisms, to refine them, to evaluate those proposed by the class. She recognizes that an algorism has little meaning for the pupil unless it is a record of his thinking. She encourages the selection of progressively more economical algorisms. She is reluctant to tell the pupil "This is *the* way to do it." She believes that concepts should come before techniques.

In summary, we like crutches which contribute to understanding. We dislike crutches which are tricky, which do not foster thinking or record thinking. We are as much concerned with learning to dispense with them as we are with having them. We know that a given crutch may be too immature for some pupils in a class and too mature for others. We know that only the teacher is able to guide the learner in his selection and subsequent rejection of a particular crutch. We deplore the fact that many teachers are oversold, and many under-sold, on crutches.

Our own thinking about crutches was significantly influenced by McConnell's chapter in the Sixteenth Yearbook of the National Council of Teachers of Mathematics, in which he states:

"Repeating the final form of a response

from the very beginning may actually encourage the habituation of immature procedures and seriously impede necessary growth.

"Intermediate steps such as the use of the 'crutch' in subtraction, aid the learner both to understand the process and to compute accurately. With proper guidance, these temporary reactions may be expected to give way to more direct responses in later stages of learning."

Should we use crutches? Yes! Choose crutches which are appropriate to the arithmetic maturity level of the learner. And equally important, discontinue them when further use limits or arrests growth in arithmetic.

Arithmetic on the March

LAURA K. EADS
Bureau of Curriculum Research, New York City Public Schools

ARITHMETIC IS GROWING in importance at all school levels. This was quite apparent at the April 1954 meeting of the National Council of Teachers of Mathematics in Cincinnati. The theme of the meeting was "Mathematics on the March" but *arithmetic* was given serious consideration by speakers and panelists representing schools, colleges, and industry.

The program included a demonstration lesson on television plus more than a dozen serious discussions of topics dealing with teaching and learning arithmetic. The following highlights of the program are presented so that teachers and supervisors who were unable to attend may share some of the values of the Cincinnati meeting.

Emerging Practices in Mathematics Education[1]

This is the title of the 1954 yearbook of the National Council of Teachers of Mathematics (Twenty-Second Yearbook, Washington, D.C., 434 pages). Practices are reported by more than 50 persons from various parts of the country, from a variety of types of school situations, and on all school levels. The yearbook is organized in six parts as follows:

Part One—Various Provisions for Differentiated Mathematics Curriculums.
Part Two—Laboratory Teaching in Mathematics.
Part Three—Teacher Education.
Part Four—New Emphasis in Subject Matter.
Part Five—The Evaluation of Mathematical Learning.
Part Six—Bibliography of "What Is Going On In Your Schools?"—1950–1953.

Learning in a Good School[2]

In a good school, teaching is defined as experimenting with learning, and learning is defined as discovery. In such a school the major purpose of education is to develop persons who find security and power in being able to confront life situations adequately and in being accepted by their fellow men. Systems of grades, diplomas, special awards, etc., are recognized as interfering with the realization of this purpose.

In a good school, distinctions are not made between content and method, general education and specialization, theoretical studies and practical studies, individual learning and social learning. All of these are considered important in a program of effective teaching and learning.

Learning in Arithmetic[3]

Differences among pupils in rates of learning arithmetic might be compared with the performance of cars of widely differing powers as they proceed along smooth open roads, and as they proceed along roads with many obstructions, such as heavy traffic. When obstructions to learning are removed, and all pupils have an opportunity to learn at a rate best for them, differences among pupils become wider and wider as they progress through school.

Teachers are reluctant to accept this principle of individual differences as it affects rates of learning. Many teachers, particularly those at upper grade levels, feel that their particular classes are "unusual" and that their pupils would not differ so widely if they had had a better background in arithmetic. Such teachers may make every effort to lessen differences among their pupils. They try to bring slow learners up to a "grade norm" and, thus, they may keep rapid learners from reaching their potential levels.

Good teaching increases differences among the learners in a class. Differences are decreased in classes where: individual attention is at a minimum, all pupils are given the same assignments, teaching aids do not challenge pupil thinking, all pupils use the same teaching aids, there are few instructional materials, arithmetic is narrowly conceived in terms of drill and computation and stereotyped problem solving, arithmetic experiences are isolated from other school or community experiences, teachers have inadequate knowledge and understanding of mathematical concepts, principles, and generalizations. In classes such as these, rapid learners are retarded and slow learners carry much heavier loads than they should.

Goals in Arithmetic[4]

The aim of arithmetic instruction should be to help pupils learn arithmetic in such a way that it may be genuinely functional for them. Mental computation and oral responses should, therefore, receive considerable emphasis.

Ways in which teachers foster discovery and learning on the part of their pupils are often as important as the content they teach. Lists of specific unrelated items to be mastered are of little value. Goals such as the following can be very helpful to teachers who see the significance of arithmetic as a system of relationships:

1. *Concepts and vocabulary* (not word-calling, however) of number and quantity; of common fractions and decimals; of processes and procedures.
2. *Understandings, principles, and generalizations* of mathematical situations which range from simple direct visual comparisons to the use of analysis, judgment, and computation when two or more variables are involved; of the number system; of number relationships; of geometric forms; of the relationship of variables.
3. *Factual knowledge and information* of how things are done and used, of important facts and definitions, of basic number combinations and processes.
4. *Skills and inferences* of arithmetical processes, of when and where to use processes, of judgments and estimates, of interpretations and inferences.
5. *Interests, attitudes, and appreciations* of mathematical organization, of assurance in place of doubt and fear, of the individuality of learning and of mathematical situations, of mathematics in hobby and vocational interests.

Content and goals should be interpreted for individual children or for groups of children in terms of: age, maturity, experience, and environment of the children; as well as the inherent nature,

structure, and difficulty of the learning materials.

Arithmetic in One School System[5]

The goals of the elementary and the junior high school program in arithmetic in New York City are to foster growth in: concept development, mathematical thinking, the solving of problems from the pupils' own experiences, skill in computation—with and without the use of paper and pencil.

Teaching and learning in arithmetic develop from the concrete to the abstract through steps called: Developmental Levels of Learning. Arithmetic is developed systematically from the first grade on. After its introduction, a topic is reinforced year after year on higher levels of concept development. Thus, growth in understanding is provided for as pupils become more mature and as they proceed through the grades. Grade placement in arithmetic thus takes on new meaning.

Articulation between the Elementary School and the Junior High School[6]

The philosophy of education and the goals in teaching mathematics should be the same for the elementary school and the junior high school. This requires that curriculum planning and the development of the mathematics curriculum be conceived as a Kindergarten through Grade 12 program.

Differences in arithmetic ability are very wide for pupils entering junior high school. Differences are widest where pupils are helped to work at maximum levels of ability in the elementary school. Some seventh grade pupils are able to deal with simple whole numbers only; others use complicated fractions, decimals, and per cents with efficiency and understanding.

Where possible, homogeneous grouping for mathematics teaching is recommended at the junior high school level. In many junior high schools it is still a new idea to divide a class into groups for mathematics teaching and learning.

The quality of homework at the later elementary and the junior high school levels needs to be carefully evaluated. Class assignments and materials used by pupils also require more careful consideration in many schools.

Some bright, more average, and many slow pupils at upper grade levels prefer to learn by rote methods, rather than by discovery. They say that this is the method they are "used to." Teachers should rethink and appraise: their methods of teaching pupils at varying levels of ability, their aims or goals for mathematics teaching, their own understanding of mathematical concepts and principles.

Junior high school teachers, generally, know less about the needs of the individual pupils in their classes than elementary school teachers. On the other hand, junior high school mathematics teachers, generally, have had a greater opportunity than elementary school teachers to learn the concepts and principles underlying mathematics. Rather than deplore inadequacies in their pupils' understanding, junior high school teachers should start their mathematics teaching where their pupils *are*, not where they think they *ought to be*. They should consider it their responsibility to develop basic concepts and understandings in elementary arithmetic where these are needed. Teachers should recognize that nothing succeeds like success and that all too many pupils accumulate failure in mathematics as the weeks, months, and years pass.

Arithmetic should be a unifying thread for all mathematics teaching throughout the junior high school. An understanding of concepts in arithmetic is basic to understanding concepts in algebra. Furthermore, a socially competent person is one who is able "to think and act intelligently in the many arithmetical situations that arise in social, cultural, and economic activities."[4]

Supervision of Mathematics Teaching[7]

Among the procedures discussed were: meetings and conferences with new teachers, individual teachers, small groups of teachers, departments, etc.; workshops for the in-service education of teachers at which participants determine problems to be studied, evaluate the workshop periodically, and study problems in advance of sessions; participation of teachers in the preparation of curriculum materials, and in the preparation, analysis, and discussion of the results of tests; demonstrations by supervisors, individual teachers, and groups of teachers in classrooms, at meetings, on radio broadcasts, on television programs; pilot classes and schools for developing procedures and materials; meetings of local and regional mathematics associations and clubs planned to meet the needs of teachers.

Research in Mathematics Education[8]

The National Council Committee on Research reported its work and planned for the presentation of some recent research and research now under way.

"Mathematics Education Research Studies—1952" is reported in: Aids for Mathematics Education, Circular No. 377, July 1953; U.S. Department of Health, Education, and Welfare; Office of Education, Washington 25, D.C. (25 p. Free) Of the 57 studies reported, 28% deal with arithmetic, from the primary grade level through high school and college levels.

Some findings of a study of the extent to which arithmetic is taught with meaning (rationalized) in one state (New Jersey) follow: the newer elementary arithmetic textbooks tend to emphasize pupil discovery, teachers tend to follow textbook procedures closely, teachers in Grades 3 and 4 are more likely to develop meaning than teachers in Grades 5 and 6, addition and subtraction are more likely to be taught with meaning than multiplication or division with whole numbers or with fractions.

Studies of specific procedures in arithmetic teaching are often inconclusive. This may be due to the brief periods usually allotted for such investigations, procedures used for evaluating learning, the nature of the problems studied, the experimental conditions set up for study.

Types of research difficult to plan and interpret but very much needed are studies in: concept formation in mathematics, appreciation in mathematics, the place of mathematics in various types of curriculum programs, sequences in learning mathematics.

Some Contributors and Topics

Thirty-Second Annual Meeting of the National Council of Teachers of Mathematics April 20–24, 1954

[1] *Presentation of the Twenty-Second Yearbook.* John R. Clark, Chairman, Teachers College, Columbia University, New York, N. Y.; Philip Peak, Indiana University, Bloomington, Ind.; John Kinsella, New York University, New York N. Y.; Joy Mahachek, State Teachers College, Indiana, Pa.; Veryl Schult, Wilson Teachers College, Washington, D. C.

[2] *Teaching and Learning in a Good School.* Donald P. Cottrell, Dean, College of Education, Ohio State University, Columbus, Ohio.

[3] *Rate of Progress in Learning Arithmetic.* Esther J. Swenson, University of Alabama, University, Ala.

[4] *Content and Goals in Arithmetic Education in Elementary School (with emphasis on grades five and six).* Ben A. Sueltz, State Teachers College, Cortland, N. Y.

[5] *Developmental Mathematics in the New York City Schools.* Laura K. Eads, Bureau of Curriculum Research, Board of Education, New York, N. Y. (A brief description of the program is given in the Twenty-Second Yearbook of the National Council of Teachers of Mathematics.)

[6] *Closer Articulation Between the Elementary and the Junior High School.* Herschel E. Grime, Directing Supervisor of Mathematics, Public Schools, Cleveland, Ohio.

To be Effective, Guidance and Counseling Programs in Mathematics Must be Continuous and Comprehensive Through the Grades. C. L. Thiele, Director of Exact Sciences, Public Schools, Detroit, Mich.

Helping Junior High School Pupils Explore Mathematics. Donald W. Lentz, Parma Schaaf Junior High School, Parma, Ohio.

Concepts From Arithmetic Which May Be Used in Teaching Fractions in Algebra. Edith Treuenfels, Putney School, Putney, Vt.

Practical Mathematics Is Challenging to Students. Frank G. Lankford, Jr. University of Virginia, Charlottesville, Va.

Determining Algebraic Ability Through Eighth Grade Mathematics. Lela Cobb, Junior-Senior High School, Wellington, Kan.

[7] *Better Supervision of Mathematics Teaching.* Mary A. Potter, Chairman, Consultant in Mathematics, Racine, Wis.; Herschel Grime, Mathematics Supervisor, Cleveland, O.; Geraldine Kauffman, Supervisor of Mathematics and Science, East Chicago, Ind.; Eugene F. Peckman, Senior Supervisor, Science and Mathematics, Pittsburgh, Pa.; Louis F. Scholl, Supervisor of Mathematics, Buffalo, N. Y.; Veryl Schult, Director of Mathematics, Washington, D. C.

A Junior High School Workshop for Mathematics Teachers. Mary A. Potter, Consultant in Mathematics, Public Schools, Racine, Wis.

Taking the Drudgery out of Problem Solving. Alice M. Hach, Slauson School, Ann Arbor, Mich.

[8] *Research in Mathematics Education.* Howard F. Fehr, Teachers College, Columbia University, New York, N. Y.; Henry Van Engen, Iowa State Teachers College, Cedar Falls, Iowa; Kenneth E. Brown, Specialist for Mathematics, U. S. Office of Education, Washington, D. C.; George McMeen, State Teachers College, Newark, N. J.; Lyman C. Peck, Iowa State Teachers College, Cedar Falls, Iowa; Maurice Hartung University of Chicago, Chicago, Ill.; Nathan Lazar, Ohio State University, Columbus, Ohio; John Kinsella, New York University, New York, N. Y.; Myron Rosskopf, Teachers College, Columbia University, New York, N. Y.; F. Lynwood Wren, George Peabody College for Teachers, Nashville, Tenn.

Textbooks Received

The Teaching of Arithmetic, second edition, Herbert F. Spitzer, Houghton Mifflin Company, 1954. 416 pages, $3.50.

This is a revised edition of Professor Spitzer's earlier text (1948) which is well known to many teachers and students. The new edition arrived too late for competent review in this issue. A later issue of THE ARITHMETIC TEACHER will carry a review.

Arithmetic for Today, Thomas J. Durell, Adaline P. Hagaman, and James H. Smith, Charles E. Merrill Books, 1954. 316 pages, 99¢ net, grades 3–6, also grades 7–8. Teacher's Manuals available, 37¢ net.

Don't Let that Inverted Divisor Become Mysterious

Edwin Eagle
San Diego State College

By pushing students too rapidly to the level of manipulation with abstract symbols we often make mysterious and meaningless for them many processes which should seem understandable and sensible. Division by a fraction is a classical example of this situation. "Invert and multiply" becomes a mumbo-jumbo that gets answers, sometimes correct ones, but which the pupils often put in somewhat the same category as rabbits pulled out of hats.

There are several methods of dividing by a fraction, and various ways of helping students rationalize these methods. Inverting the divisor and multiplying is the method most commonly used. Fortunately, this method can be put on a basis of clear understanding, though a surprisingly large number of teachers consider this a procedure which students must accept as one of those mysterious things that "just works out right but we don't know why."

Dividing by a Unit Fraction

To prevent this state of mind from developing, the process of division by a fraction should be originally presented and frequently reviewed, through grade twelve and beyond, on a basis somewhat as follows. Holding a string which is four inches long before the pupils, ask: "How many half-inch pieces can be cut from this string which is four inches long?" The majority of pupils from fifth grade up will be able to answer this, but most of them will not think of it as an illustration of the process of division. Most of them will have thought: "There are two half-inch pieces in one inch, so there must be four times two, or eight half-inch pieces in four inches. I multiply four by two to get the answer."

Other similar examples involving division of integers by common fractions should then be used. Students should illustrate the examples by such activities as marking off the fractional parts on pieces of paper, or by counting on the ruler the number of half-inch pieces in four inches. No mention of the word "division" need be made up to this point.

After some practice of this kind, with various types of applications, questions such as these may be put to the pupils: "How many two-inch lengths of string can be cut from a string which is six inches long?" "How many three-inch pieces from twelve inches?" "How many two-inch pieces from twenty inches?" These will be recognized as division problems. Attention should be focused on the fact that they are division problems. The point may then be made: "The mathematical process of finding how many times a small piece is contained in a larger piece is called division." (This of course is a restricted, not an inclusive definition of division; but it is the essential point for our immediate purpose.) $6 \div 2 = ?$ means "How many 2's are contained in 6?" On the ruler, or with objects, we can verify that there are three 2's in 6. We may write this down: $6 \div 2 = 3$.

one phase of divis

Similarly, we can ask the question, "How many half-inch pieces in four inches?" or "How many halves in four?", by writing $4 \div \frac{1}{2} = ?$ This also is division. We know that the answer is 8, for we can cut the four-inch string into half-inch pieces and by counting find that there are 8 pieces. We have not manipulated symbols to get the answer. $4 \div \frac{1}{2} = 8$ is merely a concise way of writing down the fact that we have observed by counting the pieces of string.

Now we may ask the pupils, "How do you know that there are eight half-inch pieces in four inches?" "How could you figure it out without cutting a string in pieces, or counting on a ruler?" With some repeating and rephrasing of the question most of the pupils can answer that there are two half-inch pieces in one one inch, so there must be four times two, or eight half-inch pieces in eight inches. This answer, note carefully, leads directly to an explanation of the rule: "To divide by a fraction, invert the divisor and multiply." When we invert $\frac{1}{2}$ we get 2/1 or 2, which is the number of half-inch pieces contained in one inch. That is what it must be, for this symbol, $\frac{1}{2}$, we have named "one-half," and "one-half" is the name we have given to a piece of such size that it takes 2 such equal pieces to make 1 complete unit. Now if it takes two such pieces to make one unit, it must take four times two, or eight such pieces to make four units. Hence, $4 \div \frac{1}{2}$ must equal 4×2, or $4 \times 2/1$ which is 8. Inverting the divisor and multiplying gives us the correct answer.

Similarly 1/3 (one over three) we have named "one-third," and "one-third" is the name we have given to a piece of such size that it takes three such equal pieces to make one complete unit, or one. When we invert 1/3 getting 3/1 or 3 we therefore necessarily have found how many times one third is contained in one. This is nothing mysterious; it is merely a matter of definition. It comes directly from the meaning we assigned to the symbol when we first used the symbol. Knowing that there are three one-thirds in one, we also know that there will be three times as many thirds in any number as there are units or ones in that number. That is, inverting the divisior and multiplying will give us the correct result.

Dividing by other fractions with one for a numerator may then be considered. If a ruler is used for a visual teaching aid, fourths and eights may be considered before thirds and fifths. With any fraction having one as the numerator it is evident, on the basis described above, that when we invert the fraction we find the number of times that it is contained in one unit. At this stage attention must be focused on the question, "What do we accomplish by inverting the divisor?" And the students must clearly see that the answer is, "We find how many times the divisor is contained in one, or in unity." Knowing the number of times the divisor is contained in one unit, we would multiply by two to find how many times it is contained in two units; by three to find how many times it is contained in three units, etc. The rule, "To divide by a fraction, invert the divisor and multiply" is therefore just a simple, straight-forward statement of what we see is sensible on the basis of the meanings we have given to these little two-story symbols that we call fractions, and to the words, multiply and divide.

Dividing by Non-Unit Fractions

Extending this line of analysis to division by fractions with numerators other than one is only slightly more difficult. Consider the example, $3 \div \frac{3}{4} = ?$ By counting on a line which is three inches long marked off into three-fourths-inch segments, or by counting segments on a ruler, the student can see that there are four three-fourths-inch segments in three inches. We can write this fact down in the form: $3 \div \frac{3}{4} = 4$.

To understand how the rule for division applies we note that inverting the divisor, $\frac{3}{4}$, gives us 4/3 or 1 1/3, which is the number of times that the divisor, 3/4, is contained in one unit, in this case one inch. We can look at the ruler and see that this is so. In one inch there is one $\frac{3}{4}$ inch piece and part of another such piece left over. This left over piece is one-third of a $\frac{3}{4}$ inch piece. That is, there are one and one-third of these $\frac{3}{4}$ inch pieces in one inch. This 1 1/3 or 4/3 which we get by inverting the divisor is the number of times the divisor is contained in one inch or one unit. In three inches there will be

three times as many of these $\frac{3}{4}$ inch pieces as there are in one inch. We can write this symbolically: $3 \div \frac{3}{4} = 3 \times 4/3 = 4$.

In all of the above examples, by counting pieces of string or segments on a marked line or on a ruler, it can be made clear that by inverting the divisor we find how many times the divisor is contained in one; to find how many times it is contained in some other number, we multiply this inverted divisor by that other number. Thus we see that the process of counting the number of pieces on the string or marked line or ruler not only gives us the answer, but it also helps us visualize and understand the rule for carrying out the operation.

Effective Use of Visual Aids

In a letter to Eratosthenes about 225 B.C. Archimedes wrote about "how certain mathematical questions may be investigated by means of mechanics," and he mentioned that " . . . much that was made evident to me through the medium of mechanics was later proved by means of geometry." This might suggest that in much of our teaching of mathematics we should use simple, concrete illustrations and applications, and should have students go through the mechanical operations such as cutting into pieces and arranging, measuring with the ruler, or drawing diagrams. And we should realize that these are more than motivating devices; they are more too than means of demonstrating that the method works or that the answer is correct. If we will make them so, they can be effective means of helping students better visualize and understand the essential nature of the mathematical processes involved.

"One, Two, Button My Shoe"

H. Van Engen
Iowa State Teachers College, Cedar Falls, Iowa

In the eighteenth century the child was taught the alphabet as a first step in learning to read. Such verse as:

A, in Adam's fall[1]
we sinned all.
B, thy Life to mend
this Book attend.
C, the Cat doth play
and after slay.
D, a Dog will bite
a thief at night.

helped him to memorize the sequence. In case this failed, educators recommended making the letters out of gingerbread. The child was permitted to eat each gingerbread letter he recognized.[2] Such practices probably explain why the dullards left school. They starved before they could learn to recognize enough letters to keep them alive.

After mastering the alphabet the learner was introduced to the syllabarium, which consisted of a list of syllables such as ba, be, bi, ab, ac, ad, af, etc. Syllables were eventually combined to make words and then began the real business of reading.[3]

Of course, this sounds fantastic to the modern teacher. Everybody knows that reading should not be taught by a rote memorization of the alphabet. The psycology motivating such practices is outmoded. Children learn to read by recognizing *whole* words and even short phrases. Later the alphabet comes into its own, but the ordered sequence of letters is not essential, or desirable, as a means for taking the first steps in reading.

This is an eighteenth century approach to reading; it and the psychology motivating it have been forgotten in the elementary school. But wait! Let's not be so sure. How do we teach counting in the kindergarten and first grades? That is, to those kindergartners and first graders who do not know how to count.

One, two, Button my shoe.
Three, four, Shut the door.
Five, six, Pick up sticks.
Seven, eight, Shut the gate.
Nine, ten, A big fat hen.

Isn't this the eighteenth century reading approach applied to arithmetic? Isn't this rote memorization of verse to get the number sequence in mind? Evidently the eighteenth century psychology which motivated reading is still with us in arithmetic. Rote counting corresponds to alphabet memorizing. This practice in arithmetic is not outmoded. Many methods books still advocate rote counting. This is so well known by all kindergarten teachers who have read a few methods books that there is no need for direct quotation.

Teaching Children to Count

But how can one teach children how to count without teaching them the sequence of number words by rote? It is the purpose of this paper to suggest another method which may prove to be of interest to the kindergarten and first grade teachers reading this magazine.

More and more teachers are seeing that the recognition of groups is a fundamental part of arithmetic instruction in the early grades. Evidence for this can be found in many places. Textbooks and methods books are placing more emphasis on the idea of a group than they did twenty years

[1] Taken from the 1729 edition of The New England Primer.
[2] Nila Banton Smith. *American Reading Instruction*, (New York, Silver Burdett and Company, 1934), p. 7.
[3] Ibid. P. 33.

ago. This seems a sensible procedure. Roughly, this emphasis corresponds to the practices of the reading teacher. She teaches the recognition of groups of letters before trying to take the groups of letters apart. This however, is only a very rough analogy, but it may serve to get the reader in the proper frame of mind for what follows.

Let us assume that we have a group of children who do not know how to count. What should be their first experiences with numbers?

It would seem that work with groups of objects should be basic to any first experiences with numbers. If this is so, then the identification of groups is the first step in learning to count. (Of course, comparison of groups is a prior skill.) Thus the child may first learn that this group

□
□□

is called three. He does not count. He simply learns that the sound "three" is associated with this group, in the same way that he learns that the sound "horse" is associated with a particular animal. It would undoubtedly be important to show this group of three to the child in exactly this form for the first few exeriences with the group of three. Later, he should meet the group of three arranged as

                                   □□
□□□ or □□   □ or   □

His first experiences with the group of four should probably find the objects arranged as

□□
□□

Variations of this form will be introduced as the child becomes familiar with the particular pattern of four which has been used for the first experiences with the four-group.

Assume that the child can recognize the first five-groups, namely

□   □
□□ □□□□ □ □□
□□   □□

The teacher then "engineers" the class situation so that the children notice that a group of three can be changed to a group of four by putting one more block with the three group. Continued study of this group interrelationship shows the child that the "adding one" property enables him to order them in a quantity sequence. The teacher then suggests, "Let's put them in order, with the smaller group first." The teacher, or child, may then take the one group and say, "This is one," Then taking the two-group in hand, "This is two," and so forth through the first five groups. What the child then sees is this sequence of groups.

                  □
      □ □□ □□
□ □□ □□ □□ □□

This may be a possible procedure for the first five groups but many children cannot recognize the totality of groups larger than five, if that large. This however is a generalized statement.

With certain programs of instruction this may be true of unarranged groups. By means of taking advantage of number patterns during first experiences with numbers, children certainly can learn to recognize groups in the following patterns or ones similar to these and, with systematized instruction, unarranged groups.

                        □
   □ □□ □□□ □□
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Having developed the ability to recognize groups it is now possible to continue the counting process by taking hold of each group and ordering them in their usual counting sequence. This procedure will not be developed in detail but is similar to that for the first five groups.

What Are the Advantages of this Procedure?

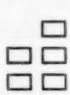
1. It replaces the rote memorization of a series of meaningless words with the association of names for groups. The

"names for groups" procedure is always the first step in learning on a very primitive level. Children learn to identify all objects by a procedure similar to that described for groups.

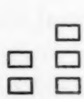
2. It avoids the very common difficulty of confusing the ordinal (first, second, third, etc.) idea with the cardinal (three in all, five in all, etc.) idea of number. What kindergarten teacher has not been perplexed and plagued by the fact that Johnny holds on to the fifth object, while counting a group of five, thereby giving clear evidence that he is thinking "fifth" when the teacher wants him to think "five." In the procedure outlined above the child takes hold of five blocks and says, "This is five," or "Five." The situation is so set up that the child cannot confuse the two ideas.

3. It introduces the essential idea of order as soon as the child can recognize his groups. The child does not learn that the nonsense words he has memorized have some reason for the order. In this case he sees the reason for the order while learning the order of the number words.

4. A foundation for future number work has been laid if the child can recognize groups. Notice what happens if the child can recognize this pattern for five



and then the group is separated in this manner



The group is still five but it has changed its appearance so that it brings to mind a group of two and a group of three. Here then is the foundation for a meaningful learning of addition combinations.

What are the disadvantages of the proposed method of teaching counting?

1. It does not conform to the pattern that the parents use at home. However this need not be a cause of concern because it hardly seems possible that there can be a conflict if the child learns a part of the number sequence by rote and then shifts to the group method.

2. It places the burden on learning the names of groups at an early stage in arithmetic instruction.

General Remarks on Teaching Counting

The author of this article has observed many teachers as they teach children how to count and he has been surprised at some of the pitfalls into which teachers often fall. For example: First counting experiences consist of counting chairs, children and other large objects. This procedure sets the stage for a confusion of the two number ideas. By the time the child counts the eighth child, the other seven children are out of his "eye span" and probably "out of his mind." Therefore he holds on to the eighth child and says, "Eight". Is it any wonder that there is a confusion? The experience has not been properly "engineered" so as to make the cardinal (group) idea of number predominants. If the first experiences involve counting small objects so that the child can fix the entire group within one "eye span" he may be able to get the group idea of "eight" and not the order idea of eighth. Large objects should not be used in first counting experiences. Large objects can be used during the first experiences with the idea of ordinal (order) number.

In conclusion, no evidence can be cited for making groups predominant in first number experiences. This article is written with the thought that some teacher may wish to investigate this method of teaching counting. It seems to make sense.

Larry and the Abacus

Orville Jenkins
Fort Miami School, Maumee, Ohio

"Right, Mr. Hale!" excitedly exclaimed the radio announcer. "You've won the granting of your lifetime wish. Yes, you're going on a tour of Egypt starting next week, all expenses paid."

"Oh, Larry," Jean asked her brother enviously, "what would you have wished? I'd like to go to Egypt too, but I don't believe I could ever decide on just one best wish."

"That would be easy enough for me," Larry yawned. "I'd be satisfied to be living a long time ago when no one had to study arithmetic." He held up his book which he had been studying halfheartedly. "I can't do these long addition problems correctly. They put me to sleep. Give me the Middle Ages we studied in history class. I'll bet King Arthur's knights didn't have to study arithmetic. That would be the life! I'd just ride around on my horse rescuing fair laides and fighting in silver armor at tournaments—"

"Larry, be sensible! Larry, Larry, wake up!" For the book had slipped from Larry's hands and his eyes were closed. Little did Jean realize what was going through his dreaming mind:

"I dub thee Sir Larry," King Arthur was saying while giving Larry the traditional blow on the back of the neck to prove his courage. "Now for your task, Sir Larry. We are in sad need of silver and gold. Thieves have stolen the royal treasury from the underground vaults. It is believed the villians live in the Black Forest. Your duty, Sir Larry, is to return this money."

"I shall be pleased, my lord, to find and return the money to you—else you shall never see this face again."

As Larry dreamed, he saw himself riding into the Black Forest and fearlessly scattering the few thieves guarding the money. The royal coins and gems were lying here and there in heavy bags which chinked prettily as Sir Larry heaved them over his shoulder and tied them to his white steed. Only one matter disturbed him. Had he collected all the money? He had no idea how much was there, and had not dared look too long in the thieves' den for fear the rest would return.

Once back in King Arthur's court, Sir Larry placed the coin bags at the feet of the king, then knelt.

"I have returned, my lord, with the royal money. I only hope I have all of it. I had no way of knowing whether some of it was buried or on the person of another thief."

"Well done, Sir Larry. You shall be richly rewarded. Cease your worry. I shall call in my royal treasurer to count the money. He will have an account and can tell you whether all has been returned."

Larry's eyes grew large as he saw the royal treasurer enter without paper and pen, but carrying a strange wooden frame holding rods of beautiful marble beads. Was the man insane that he played with beads?

"Here is the money, royal treasurer. But how will you count it?"

"With my abacus, of course," answered the treasurer, holding up the wooden frame. "How else?"

Larry was about to mention paper or an adding machine, but thought better of it. He had better watch before saying too much, for the royal treasurer already seemed to think he was stupid.

"Please empty the bags on the counter, Sir Larry," demanded the treasurer.

The counter? wondered Larry. This wasn't a store.

"Here on the table, Sir Larry," the treasurer pointed impatiently, "where I can count the money."

"Oh," Larry thought, "that's where we got the modern word *counter*. It was where the money was handed over and counted in a store."

After he poured the coins on the table Larry saw the treasurer move the beads on the frame to the left, the lower beads for smaller coins, and the upper beads for the more valuable coins. He wrinkled his forehead and scratched his head, but he couldn't quite understand how the treasurer was counting. Then the system began to dawn on him. Each bead stood for a certain number of units. The bottom row of beads each stood for one, the beads on the next row for ten, the next for a hundred, and so on. After so many of the lower beads were used, the number could be carried to an upper bead of more value.

Many, many hours later the counting was finished and Larry was relieved to hear:

"Every cent taken from the royal treasury is here, Sir Larry. I congratulate you. My fine abacus gives an accurate count."

After the congratulation Larry felt he could afford to sound a trifle foolish and dared to ask, "Tell me, please, sir, is the abacus a new counting method?"

"New? Ha ha, that's a good one!" The treasurer was not a very understanding man. "It has been used almost as long as civilized people have existed. Why, the Egyptians first recorded numbers on an abacus that used small stones. The Romans also used small stones called *calculi*, so that's where we get the word calculate meaning to count or figure. The Greeks and Babylonians had abacuses too, and the Chinese and Japanese use them, I've heard. The Chinese call theirs the *suan pan*, and some of the Chinese use their counting boards so expertly and swiftly you and I would be astonished. Instead of a book and paper, a Roman schoolboy would carry his arithmetic board or abacus and a box of stone counters to school. If his parents were very wealthy, his counters were ivory dice or small silver coins."

"That is very interesting," Larry continued, "but can you do anything besides addition on your abacus?"

"Of couse. I can subtract, divide, and multiply. It is much better for counting, however."

"But why don't you write your numbers?" Larry asked. "Wouldn't that be much quicker and easier?"

"Write them? What do you mean? Of course we write the day and year or a page number with Roman numerals. You know them, don't you? Every knight ought to." He wrote with his fingers in the air: I, II, III, IV, and V.

"Yes, I know about them," Larry laughed, "but they are such funny figures. They don't have many curves in them, and it takes so long to write a big number."

"Of course they are almost all made of straight marks," the treasurer replied. "It is much easier to carve straight marks into stone tablets, and that's what the Romans first used for writing that would last."

"But why don't you use numbers like these?" Larry traced with his finger in the light dust on the table, "1, 2, and 3? You can do all sorts of things with them."

"Oh yes, I have seen them used once." The treasurer was really a learned man. "A few scholars say we should all use them. They are called Arabic figures, I believe, and are a new system. However, I have recently talked to a merchant who traveled to far-off Arabia, and he said they do not use these new figures there, but that they are from India, and a clever Arabian merely learned of the system and wrote a book about it."

"You are certainly a wise man." Larry was amazed that the royal treasurer knew more about the history of the numbers used every day in school than he did. He

had supposed our numbers always existed and hadn't thought that someone had to invent them. To think the numbers had come from Arabia or India, the lands of tents, deserts, mountains, and idols! Perhaps the treasurer could tell him more. He thought of the long columns of figures in his own arithmetic book. "Do the scholars find the new numbers hard to learn?" Larry asked.

"No, not difficult, but quite different, although a lot of fun to play with in the mind. The numbers I saw briefly had a round 0 which stands for nothing and is confusing."

"That's a zero. But why is a zero confusing?"

"Isn't it silly to have a number which stands for nothing? The Roman numerals have no such number and need none. But it is part of the Arabic system. As if there isn't enough in the world to learn besides a mark that stands for nothing!"

Larry was disturbed. There must be a good reason for a zero. Otherwise how could you tell one from ten or one hundred? But to explain it seemed impossible. Perhaps he had better change the subject before the treasurer asked him more embarrassing questions. "Have there ever been any other written numbers?"

"Yes, many. Have you ever head of hieroglyphics? They were pictures the Egyptians used to represent words and numbers. A picture of a staff was one, a heelbone was ten, a scroll was one hundred and a lotus flower, one thousand. Ten staffs were a heelbone, ten heelbones made a scroll, and ten scrolls made a lotus flower. Of course there were pictures for numbers much higher. The Egyptians must have been clever with arithmetic or they could never have built their fine pyramids."

"But why were ten of everything always used? Why didn't five staffs make a heelbone? How did it happen that Egyptian counting was so like ours? They lived long ago, and we don't use their word pictures any more."

"There's a reason for it. Sir Larry, my boy." The treasurer was becoming more kindly now. "How many fingers do you have?"

"Why, ten, of course."

"And how did you count when you were very small?"

"With my fingers, I suppose."

"That's right. And so have all other persons, even the most educated. In fact, the Roman numeral I is supposed to stand for one finger or *digitus*. That's why the word "digit" now means finger or number. And according to some stories the Roman V is supposed to be a picture of one hand, and a X is two arms crossed, which would be a total of ten fingers. Did you know, too, that you can show almost any number you want with your fingers?" He demonstrated by bending down the little finger of his left hand. "What number would this be?"

"That's easy. One. But how would you count over ten? Take off your shoes?" Larry laughed at the idea.

"That really isn't a joke! Some of the African natives and others who naturally go barefoot do exactly that—count with their toes too. But there is another way. See this!" He bent his left thumb down far on his palm. "This is fifty, and there are other signs for the rest of the numbers."

"But don't you run out of combinations?" Larry wondered.

"Not at all. You see the right hand is used for the hundreds, and the left hand again following the same pattern as the lower numbers for the thousands."

"My, it would take forever to learn! I'm glad I don't have to."

"Yes, it does take time, but many people including early Greeks have used this, some still do, especially savages. There have been many number systems up until our King Arthur's time. But some get discarded when new and better ones came along."

"Then why don't you learn the new Arabic system? It is simpler, and you could toss away your abacus."

"Get rid of my abacus? Man will always use it, of course. Without it I would have forgotten the count many times before I had finished."

"But why don't you put the number on paper every once in a while and add later? That's a lot easier for big numbers than an abacus. Or why don't you—"

Larry's head jerked. Someone was tugging at his arm. "Wake up, Larry." It was Jean. "Larry, it's nine o'clock, and you know we promised Mother and Dad to go to bed at this time. By the way, what's that word you were muttering about while your were waking? Ab—abacus?"

"Abacus? Oh, now I remember the dream. It's an—oh, never mind now. I'll tell you tomorrow morning."

Sisters, thought Larry with disgust, don't know anything. He could never explain to her about our long addition and how much simpler it was than an abacus or other system for big numbers. It took a man to understand arithmetic and how it had improved. Larry patted his book and started upstairs.

Editors' Note

Several plays and stories dealing with some aspect of arithmetic have been received by THE ARITHMETIC TEACHER. The story of "Larry and the Abacus" illustrates a sixth grade teacher's effective way of whetting the interest of his pupils in the history of numbers. In dramatizing this story, Mr. Jenkins correlated arithmetic and social studies. His pupils developed new interests and a keener awareness of the development of numbers and counting. The editors believe that many other teachers should, with the cooperation of pupils, develop plays and stories. Much of the value comes from the "research" and writing, the final dramatization is a fitting memorable climax. Do not reproduce the story of Larry, rather have your class create a story of its own. Was King Arthur's coinage on the base ten?